CONVEXITY OF THE FREE BOUNDARY FOR AN EXTERIOR FREE BOUNDARY PROBLEM INVOLVING THE PERIMETER

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Abstract. We prove that if the given compact set K is convex then a minimizer of the functional

$$I(v) = \int_{B_R} |\nabla v|^p dx + \mathrm{Per}(\{v > 0\}),\ 1 < p < \infty,$$

over the set $\{v \in H_0^1(B_R)|\ v \equiv 1 \text{ on } K \subset B_R\}$ has a convex support, and as a result all its level sets are convex as well. We derive the free boundary condition for the minimizers and prove that the free boundary is analytic and the minimizer is unique.

1. Introduction

1.1. **The Problem.** The following problem has been considered in [Maz]: given a bounded domain $K \subset B_R \subset \mathbb{R}^n$ (R large), satisfying the interior ball condition, find a (local) minimizer of the functional

$$I(v) = \int_{B_R} F(|\nabla v|)dx + \mathrm{Per}(\{v > 0\}) \tag{1}$$

over the set of functions $\{v \in H_0^1(B_R)|v \equiv 1 \text{ on } K\}$, where $F \in C^1([0, +\infty))$ is a positive convex function, with $F(0) = 0$ and for some $1 < p < +\infty$ and $0 < \lambda < \Lambda < +\infty$

$$\lambda t^{p-1} \leq F'(t) \leq \Lambda t^{p-1}.$$

Here we set $\mathrm{Per}(\{v > 0\}) = +\infty$ if $\chi_{\{v>0\}} \notin BV(\mathbb{R}^n)$. This problem is the one-phase exterior analogue of the problem introduced in [ACKS] for a functional with general convex function $F(t)$ in the first term (in [ACKS] they treat the case $F(t) = t^2$).

In [ACKS] (two phase, $p = 2$) and [Maz] (one phase, $1 < p < \infty$) it is proved that the minimizers are Lipschitz continuous. This gives that the free boundary $\Gamma_u := \partial\{x|u(x) > 0\}$ is an almost minimal surface, thus $C^{1,1/2}$-smooth. Let us recall some facts from the theory of almost minimal surfaces following [T].

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A set Ω has almost minimal boundary in B_1 if for every $A \Subset B_1$ there exist R, $0 < R < \operatorname{dist}(A; \partial B_1)$ and

$$\alpha : (0, R) \to [0, +\infty), \ \ \alpha(r) \downarrow_{r\to 0} 0,$$

such that

$$\operatorname{Per}(\Omega; B_r(x)) \leq \operatorname{Per}(\Omega'; B_r(x)) + \alpha(r)r^{n-1}$$

for every $x \in A$, $r \in (0; R)$ and Ω' with $\Omega'\Delta\Omega \Subset B_r(x)$.

Lemma 1. [T] *Suppose Ω has almost minimal boundary in B_1 with $\alpha(r) = r^{2\lambda}$, $\lambda \in (0, 1/2]$. Then*
(i) $\partial^*\Omega$ is a $C^{1,\lambda}$ hypersurface, and
(ii) $H^s(\partial\Omega\backslash\partial^*\Omega \cap B_1) = 0$ for each $s > n - 8$.

Here $\partial^*\Omega$ is the reduced boundary of Ω (see [EG]).

As it is shown in [ACKS] and [Maz] (one phase, $1 < p < \infty$) the Lipschitz regularity of the minimizer gives that the free boundary is an almost minimal surface with $\alpha(r) = r$ hence the reduced boundary $\Gamma^*_u := \partial^*\{x|u(x) > 0\}$ is $C^{1,\frac{1}{2}}$ regular and the singular set is of Hausdorff dimension $n - 8$ or less.

Remark 2. *Any blow-up at the almost minimal surface is a minimal cone (see [T], p. 85). Thats why all points of the free boundary at which we can find a supporting smooth surface belong to the smooth part Γ^*_u.*

In this paper we restrict ourselves to the case $F(t) = t^p$, $p > 1$, i.e., the functional

$$I(v) = \int_{B_R} |\nabla v|^p dx + \operatorname{Per}(\{v > 0\}), \tag{2}$$

though we want to mention that the same ideas and methods will work in the general case (1) if we put some additional (rather weak) conditions on the function F.

The main result of this paper is the following theorem.

Theorem A. *If $K \Subset B_R$ is a convex set with non-empty interior and u is a minimizer of (2) over the set $\{v \in H^1_0(B_R)|v \equiv 1 \text{ on } K\}$ then the minimizer is unique, the set $\{u > 0\}$ is convex and the free boundary $\partial\{u > 0\} \cap B_R$ is an analytic surface.*

We also derive the free boundary condition in case of general (non-convex) K and prove that any minimizer u_K satisfies the following inclusion

$$\Omega_{u_K} \subset \Omega_{u_{\operatorname{cov}(K)}},$$

see the notations below. This means for instance that for large R we will have $\Omega_{u_K} \Subset B_R$.

It is noteworthy that convexity results for the so-called Bernoulli free boundary problem has been extensively studied, see [A], [HS1], [HS2] and the references therein.

1.2. **Notations.** In the sequel we use the following notations:

$\mathbf{R}^n_+$	$\{x \in \mathbf{R}^n : x_1 > 0\}$
$B(z,r)$	$\{x \in \mathbf{R}^n : \lvert x - z\rvert < r\}$,
B_r	$B(0,r)$,
χ_D	characteristic function of the set D,
∂D	boundary of the set D,
Ω_u	$\{x \in \mathbf{R}^n : u\,(x) > 0\}$,
Γ_u	$\partial\Omega_u$ the free boundary,
Γ^*_u	$\partial^*\Omega_u$ the reduced boundary of Ω_u (see [EG]),
$\operatorname{cov}(U)$	the convex hull of the set U.

1.3. **Organization of the paper.** In Sections 2 and 3 we develop some technical tools which will be used in the proofs coming after. In Section 4 we derive the free boundary condition and prove that the reduced free boundary is analytic. An interesting geometric result about the mean curvature of the boundary of the convex hull of a non-convex domain is proved in Section 5. The main result of the paper, the convexity of the free boundary and the uniqueness of the minimizer is proved in Section 6.

2. An energy estimate for p-harmonic extensions

Assume $K \Subset \Omega_1 \subset \Omega_2$, where K, Ω_1, Ω_2 are open and bounded subsets of $\mathbb{R}^n$ with non-empty interior, and that u_j minimizes the functional

$$J(v) = \int |\nabla v|^p dx \tag{3}$$

in the class of functions $\{v \in H^1_0(\Omega_j) |\, v \equiv 1 \text{ on } K\}$ $(j = 1, 2)$. Then we say that u_2 is the p-harmonic extension of u_1 from Ω_1 to Ω_2. The use of word "extension" is a little bit misleading here, since $u_1 \neq u_2$ in Ω_1, but we keep it as an analogy to extension by zero, which would be standard in this situation, because of zero boundary data on the boundary of Ω_1. We also extend functions in $H^1_0(\Omega)$ by zero and assume that they are defined in all $\mathbb{R}^n$.

In this section we prove the following lemma.

Lemma 3. *If u_2 is the p-harmonic extension of u_1 from Ω_1 to Ω_2, where Ω_1 and Ω_2 have piecewise $C^{1,\alpha}$ boundary. Then*

$$\begin{aligned} 0 \le \int_{\Omega_2} |\nabla u_1|^p - |\nabla u_2|^p dx - \int_{\Omega_2\setminus\Omega_1} (p-1)|\nabla u_2|^p dx \le \\ - p\int_{\partial\Omega_1\setminus\partial\Omega_2} u_2[|\nabla u_1|^{p-2}\partial_\nu u_1 - |\nabla u_2|^{p-2}\partial_\nu u_2] dH^{n-1}. \end{aligned} \tag{4}$$

Proof. We write

$$v\Delta_p u = -p|\nabla u|^{p-2}\nabla u\nabla v + p\mathrm{div}(v|\nabla u|^{p-2}\nabla u),$$

and using Gauss' theorem we obtain

$$\int_{\Omega_2} p|\nabla u_2|^{p-2}\nabla u_2\nabla(u_1-u_2)dx = \tag{5}$$
$$-\int_{\Omega_2}(u_1-u_2)\Delta_p u_2 dx + \int_{\partial\Omega_2} p|\nabla u_2|^{p-2}(u_1-u_2)\partial_\nu u_2 dH^{n-1} = 0$$

From here we have

$$\int_{\Omega_2}|\nabla u_1|^p - |\nabla u_2|^p =$$
$$\int_{\Omega_2}(p-1)|\nabla u_2|^p + [|\nabla u_1|^p - p|\nabla u_2|^{p-2}\nabla u_1\nabla u_2]dx$$
$$= \int_{\Omega_2\setminus\Omega_1}(p-1)|\nabla u_2|^p dx +$$
$$\int_{\Omega_1}|\nabla u_1|^p - |\nabla u_2|^p - p|\nabla u_2|^{p-2}\nabla u_2\nabla(u_1-u_2)dx.$$

Now we are going to estimate the last integral. Let us consider the following function

$$\Phi(t) = |\nabla(u_2 + t(u_1-u_2))|^p.$$

From the convexity and monotonicity of t^p it follows that Φ is convex in t. So we can write

$$0 \le \Phi(1) - \Phi(0) - \Phi'(0) \le \Phi'(1) - \Phi'(0).$$

This gives us exactly the following

$$0 \le |\nabla u_1|^p - |\nabla u_2|^p - p|\nabla u_2|^{p-2}\nabla u_2\nabla(u_1-u_2) \le$$
$$p|\nabla u_1|^{p-2}\nabla u_1\nabla(u_1-u_2) - p|\nabla u_2|^{p-2}\nabla u_2\nabla(u_1-u_2)$$

in Ω_1. Integrating partially in the domain Ω_1 like in (5) we get that

$$\int_{\Omega_1} p|\nabla u_1|^{p-2}\nabla u_1\nabla(u_1-u_2) - p|\nabla u_2|^{p-2}\nabla u_2\nabla(u_1-u_2) =$$
$$-p\int_{\partial\Omega_1\setminus\partial\Omega_2} u_2(|\nabla u_1|^{p-2}\partial_\nu u_1 - |\nabla u_2|^{p-2}\partial_\nu u_2)dH^{n-1}$$

□

Remark 4. *Note that the first inequality in (4) does not require smoothness assumptions on $\partial\Omega_i$, $i=1,2$.*

3. The Hopf lemma for p-harmonic functions in domains with Liapunov-Dini boundary

Let us present the definition of Liapunov-Dini surface following [W].

Definition 5. *A Liapunov-Dini surface S is a closed, bounded $(n-1)-$dimensional surface satisfying the following conditions:*

(a) At every point of S there is a uniquely defined tangent (hyper-)plane, and thus also a normal.

(b) There exits a Dini modulus of continuity $\epsilon(t)$ such that if β is the angle between two normals, and r is the distance between their foot points, then the inequality $\beta \le \epsilon(r)$ holds.

(c) There is a constant $\rho > 0$ such that for any point $x \in S$ any line parallel to the normal at x meets $S \cap B_\rho(x)$ at most once.

A modulus of continuity $\epsilon(r) \to 0$ as $r \to 0$ is called Dini modulus of continuity if $\int t^{-1}\epsilon(t)dt < \infty$.

Note that a domain E with Liapunov-Dini boundary satisfies a kind of interior and exterior Dini condition in the following sense: There exists a convex Liapunov-Dini domain K such that for any point $x_0 \in \partial E$ there exists a translation and rotation K_{x_0} of the domain K satisfying

$$K_{x_0} \subset E,\ (K_{x_0} \subset \mathbb{R}^n \backslash E) \text{ and } \partial K_{x_0} \cap \partial E = \{x_0\}.$$

The following lemma is proved in [J], see also [MPS].

Lemma 6. *Let $\Omega \backslash K$ be a convex ring and u be its p-capacitary potential. Then*

$$\Delta_q u \le 0, \text{ if } 1 < q \le p$$

and

$$\Delta_q u \ge 0, \text{ if } p \le q \le \infty.$$

Now we formulate and prove the main result of this section, which might be a known result but we could not find any reference.

Lemma 7. *Assume u is a p-harmonic function in the domain U. Further assume $y \in \partial U$, ∂U satisfies the interior and exterior Dini conditions locally near y and $u(x) \ge u(y)$ for all $x \in U$. Then there exist positive constants r_0, c, C such that*

$$cr < \max_{\{x:|x-y|<r\}} u(x) - u(y) < Cr,$$

for $0 < r < r_0$.

Proof. Let us take the function w to be the minimizer of the Dirichlet integral in $\{v \in H^1_0(K_2) | v \equiv 1 \text{ on } K_1\}$, where K_1 and K_2 are convex domains with Liapunov-Dini boundary and $K_1 \Subset K_2$. Thus we have $\Delta w = 0$ on $K_2 \backslash \overline{K_1}$. From the Hopf lemma for harmonic functions (Thm. 2.5, [W]) and the convexity and regularity of the level sets of w (see [L]) we know that $\nabla w(x) \neq 0$, for any $x \in K_2 \backslash \overline{K_1}$. Now we

will prove the existence of a smooth, convex function $f:[0,1]\to[0,1]$, $f(0)=0,\ f(1)=1$ such that

$$\Delta_p f(w)\geq 0$$

in $K_2\backslash\overline{K_1}$ and $0<f'(t)<+\infty$ for all $t\in[0,1]$. This will mean that the function $f(w)$ is a sub-solution for Δ_p and has non-vanishing gradient, thus it will work as a standard barrier function.

We have

$$\Delta_p f(w)=p|\nabla f(w)|^{p-2}\Delta f(w)+p(p-2)|\nabla f(w)|^{p-4}\Delta_\infty f(w),$$

where

$$\Delta_\infty v=\sum_{i,j}v_{ij}v_iv_j$$

is the well known infinity Laplace operator[1]. On the other hand

$$\nabla f(w)=f'(w)\nabla w,$$

$$\Delta f(w)=f'(w)\Delta w+f''(w)|\nabla w|^2=f''(w)|\nabla w|^2,$$

$$\Delta_\infty f(w)=(f'(w))^3\Delta_\infty w+(f'(w))^2f''(w)|\nabla w|^4.$$

So we need to find a function f such that

$$\Delta_p f(w)=pf''(w)f'(w)^{p-2}|\nabla w|^p+$$
$$p(p-2)(f'(w)|\nabla w|)^{p-4}\left[(f'(w))^3\Delta_\infty w+(f'(w))^2f''(w)|\nabla w|^4\right]\geq 0,$$

or

$$\frac{f''(w)}{f'(w)}\geq\frac{2-p}{p-1}|\nabla w|^{-4}\Delta_\infty w. \tag{6}$$

We see that for $p\geq 2$ we can take $f(t)\equiv t$. This follows from the Lemma 6.

In case $1<p<2$ we continue as follows. We have from [W] that the derivatives of w are continuous up to the boundary and do not vanish. Moreover we have bounds for the second derivatives of w near the boundary (formula (2.4.1) in [W])

$$|D^2w|\leq\zeta(d(x)),$$

where $\zeta\in L^1(0,dist(K_1,\mathbb{R}^n\backslash K_2)/2)$, and $d(x)$ is the distance function from the boundary of the domain $K_2\backslash K_1$. Coming back to our case there exists a function $\zeta_1(t)\in L^1((0,1))\cap C((0,1))$ such that $|\nabla w|^{-4}|\Delta_\infty w|\leq\zeta_1(w)$ in $K_2\backslash\overline{K_1}$. Let us now integrate (6) in $w\in[t,1]$,

$$\int_t^1\frac{f''(\tau)}{f'(\tau)}d\tau=\int_{f'(t)}^{f'(1)}\frac{ds}{s}\geq\frac{2-p}{p-1}\int_t^1\zeta_1(\tau)d\tau.$$

[1]In our definition by taking $H(t)=t^p$ the operator Δ_p differs from the usual one by a factor p.

Thus we can take for instance

$$f(t) = c \int_0^t \exp\left(-\frac{2-p}{p-1} \int_\tau^1 \zeta_1(s) ds \right) d\tau,$$

where the constant $c > 0$ is chosen to get $f(1) = 1$.

Note that the function $1 - f(w)$ is a super-solution for Δ_p in $K_2 \backslash \overline{K_1}$ and will give us bounds from above. □

Remark 8. *In case of the $C^{1,\alpha}$ boundary the existence of the gradient of the function u at the boundary is known (see* [Li]*) and we can write*

$$0 < c < |\nabla u(y)| < C.$$

4. The free boundary condition

First let us prove that for convex K the free boundary stays away from the set K.

Lemma 9. *Let the set K be convex and u be the minimizer of (1). Then there exists a constant δ depending on n, p and the set K such that dist$(x, K) \geq \delta$, for all $x \in \Gamma_u$.*

Proof. Let us take the points $y \in \Gamma_u$ and $x \in K$ such that $\text{dist}(y, x) = r_0 := \text{dist}(\Gamma_u, K)$.

Let us denote by $V(r) := |B_r(x) \backslash \Omega_u|$ and $A(r) = \mathcal{H}^{n-1}(\partial B_r(x) \backslash \Omega_u)$, so that $V'(r) = A(r)$. Then we have from the isoperimetric inequality and from the minimality condition that

$$(V(r))^{\frac{n-1}{n}} \leq c\text{Per}(V(r)) \leq 2cA(r).$$

Now integrating

$$2c \leq V'(r)(V(r))^{-\frac{n-1}{n}}$$

in the interval (r_0, r) we obtain

$$H^{n-1}(\partial B_r(x) \backslash \Omega_u) \geq c(r - r_0)^{n-1}, \tag{7}$$

where c depends on the dimension.

From the convexity of K and the fact that it has a non-empty interior we know that there is a cone $\mathcal{C}$ and $r_K > 0$ such that for any point $y \in \partial K$ there exists a rotation and translation of the set $\mathcal{C}_{r_K} := \mathcal{C} \cap B_{r_K}$ such that $0 \mapsto y$ and $\mathcal{C}_{r_K}$ is mapped into K. In other words at any point of ∂K we can put a conical set of fixed opening and length r_K lying inside K. This gives that

$$H^{n-1}(\partial B_r(y) \cap K) \geq c_K r^{n-1},$$

for $r < r_K$ and all $y \in \partial K$, c_K depends only on K. Let us take the function $\zeta(x)$ to be the p-harmonic potential of the convex ring $B_1(0) \backslash (B_{1/4}(0) \cap \mathcal{C})$.

Step 1: We first exclude the case $r_0 = 0$. Assume $y \in \Gamma_u \cap K$. We take as a perturbation of u the function $v(x) := \max(u(x), \zeta_r(x))$, where $\zeta_r(x) := \zeta((x - y)/r)$ and we can without loss of generality

assume that $\{x|\zeta_r(x)=1\}\subset K$ for all $r<r_K$. Note that the function ζ_r is the p-harmonic extension of the function u from $\{u<\zeta_r\}\cap\Omega_u$ to $\{u<\zeta_r\}$, which together with Lemma 3 and Remark 4 gives the following

$$(p-1)\int_{B_r(y)\setminus\Omega_u}|\nabla\zeta_r|^p dx \le \int_{\{u<\zeta_r\}}|\nabla u|^p-|\nabla\zeta_r|^p dx \le \\ H^{n-1}(\partial B_r(y)\setminus\Omega_u)-\mathrm{Per}(\Omega_u;B_r(y))\le Cr^{n-1}. \tag{8}$$

The second inequality uses the fact of u being a minimizer. On the other hand using (7) (remember that r_0 is assumed to be 0) we obviously can find constants c_1,c_2 depending only on K, n and p such that

$$\int_{B_r(y)\setminus\Omega_u}|\nabla\zeta_r|^p dx \ge \int_{B_r(y)\setminus(\Omega_u\cup B_{3r/4}(y))}|\nabla\zeta_r|^p dx \ge \\ c_1\int_{B_r(y)\setminus B_{3r/4}(y)}|\nabla\zeta_r|^p dx \ge c_2 r^{n-p}, \tag{9}$$

a contradiction. In the second and third inequalities of (9) we used (7) and the fact that $cr^{-1}<|\nabla\xi_r|<Cr^{-1}$ in $B_r(y)\setminus B_{3r/4}(y)$ for some positive constants depending on r_K.

Step 2: Now we know that $r_0>0$ and we can use the estimates used by Mazzone (Lemma 3.2, [Maz]) for the terms in (8). If we denote by $d'(x):=\mathrm{dist}(x,\partial B_r(y))$ we obtain that

$$H^{n-1}(\partial B_r(y)\setminus\Omega_u)-\mathrm{Per}(\Omega_u;B_r(y))\le \\ -\int_{\partial^*(B_r(y)\setminus\Omega_u)}\langle\nabla d',\nu\rangle dH^{n-1}= \\ -\int_{B_r(y)\setminus\Omega_u}\Delta d' dx\le\frac{c_n}{r_0}|B_r(y)\setminus\Omega_u|. \tag{10}$$

On the other hand as in (9)

$$\int_{B_r(y)\setminus\Omega_u}|\nabla\zeta_r|^p dx\ge c|B_r(y)\setminus\Omega_u|r^{-p},$$

where c depends only on n and K. Summing up we obtain that

$$cr_0^{-p}\le\frac{c_n}{r_0},$$

where all constants depend only on n, p and K. $\square$

Lemma 10. *The reduced free boundary* Γ^* *is analytic and* $\Gamma^*\cap B_R$ *satisfies the free boundary condition*

$$(p-1)|\nabla u|^p=\kappa(\Gamma^*_u), \tag{11}$$

where κ is the mean curvature. Moreover on $\overline{\Omega}_u \cap \partial B_R$ we have pointwise the inequality

$$(p-1)|\nabla u|^p \geq \kappa(\partial B_R). \tag{12}$$

Proof. Step 1: We first derive the free boundary condition in the weak sense using the domain variation method and show that Γ^* is $C^{2,\alpha}$ regular.

Assume the origin is a reduced free boundary point $0 \in \Gamma^*$, thus we can assume that for a small $\delta > 0$ in the neighborhood $\mathcal{N}_\delta := \{x||x'| < \delta, |x_n| < \delta\}$ the free boundary is a graph $\Gamma^* = \{x|x_n = \phi(x')\}$, with $\phi \in C^{1,1/2}$ (see Section 1.1 and [Maz]), $\phi(0) = |\nabla\phi(0)| = 0$.

For a vector field $\eta \in C^1_0(\mathcal{N}_\delta; \mathbb{R}^n)$, $\sup \eta \leq 1$ and small enough ϵ consider the bijective map $\Phi_\epsilon(x) = x + \epsilon\eta(x)$ and the function $u_\epsilon(y) = u(\Phi_\epsilon^{-1}(y))$.

From the minimality of u we have that

$$\int_{B_R} |\nabla u_\epsilon(y)|^p dy - \int_{B_R} |\nabla u(x)|^p dx + \mathrm{Per}(\{u_\epsilon > 0\}) - \mathrm{Per}(\{u > 0\}) \geq 0.$$

Let us now calculate the terms above. We are following the book of Ambrosio, Fusco, Pallara ([AFP], page 360), where all these calculations are carried out in a similar situation. Since

$$\int_{B_R} |\nabla u_\epsilon(y)|^p dy = \int_{B_R} |\nabla u(x) \cdot \nabla\Phi_\epsilon^{-1}(\Phi_\epsilon(x))|^p |\det \nabla\Phi_\epsilon(x)| dx$$

and

$$\nabla\Phi_\epsilon^{-1}(\Phi_\epsilon(x)) = I - \epsilon\nabla\eta(x) + o(\epsilon),$$

$$\det \nabla\Phi_\epsilon(x) = 1 + \epsilon \mathrm{div}\eta(x) + o(\epsilon),$$

we see that

$$\int_{B_R} |\nabla u_\epsilon(y)|^p dy - \int_{B_R} |\nabla u(x)|^p dx = \epsilon \int_{B_R} \left(|\nabla u(x)|^p \mathrm{div}\eta(x) - p|\nabla u|^{p-2}\langle\nabla u, \nabla\eta \cdot \nabla u\rangle\right) dx + o(\epsilon).$$

On the other hand

$$\mathrm{Per}(\{u_\epsilon > 0\}) - \mathrm{Per}(\{u > 0\}) = \epsilon \int_{\Gamma^*} \mathrm{div}^{\Gamma^*} \eta d\mathcal{H}^{n-1} + o(\epsilon),$$

where $\mathrm{div}^S F(x) = \sum_{k=1}^n \langle\nabla^S F_k(x), e_k\rangle$ is the tangential divergence of F on surface S and $\nabla^S f$ is the projection of $\nabla f(x)$ on the tangent space $T_x S$ (see Definition 7.27 and Theorem 7.31 in [AFP]).

Integrating by parts in $\mathcal{N}_\delta \cap \{u > 0\}$ we obtain

$$\int_{B_R} |\nabla u(x)|^p \mathrm{div}\eta(x) dx =$$

$$\int_{\Gamma^*} |\nabla u(x)|^p \langle \eta(x), \nu \rangle d\mathcal{H}^{n-1} - \int_{B_R} \langle \nabla |\nabla u(x)|^p, \eta \rangle dx,$$

where ν is the normal vector, and

$$-\int_{B_R} p|\nabla u|^{p-2} \langle \nabla u, \nabla \eta \cdot \nabla u \rangle dx =$$

$$\int_{B_R \cap \{u>0\}} \Delta_p u \langle \eta, \nabla u \rangle + p|\nabla u|^{p-2} \langle \eta, \nabla^2 u \cdot \nabla u \rangle dx -$$

$$\int_{\Gamma^*} p \langle \eta, \nabla u \rangle |\nabla u(x)|^{p-2} \partial_\nu u d\mathcal{H}^{n-1}.$$

Noting that $\langle \nabla |\nabla u(x)|^p, \eta \rangle = p|\nabla u|^{p-2} \langle \eta, \nabla^2 \cdot u \nabla u \rangle$ and summing up and letting ϵ go to 0 we obtain that

$$\int_{\Gamma^*} \mathrm{div}^{\Gamma^*} \eta d\mathcal{H}^{n-1} = \int_{\Gamma^*} (p-1)|\nabla u(x)|^p \langle \eta(x), \nu \rangle d\mathcal{H}^{n-1},$$

for any $\eta \in C_0^1(\mathcal{N}_\delta; \mathbb{R}^n)$.

If we now rewrite the left hand side in terms of function ϕ and use the Proposition 7.40 from [AFP] we obtain that

$$-\,\mathrm{div}\left(\frac{\nabla \phi(x')}{\sqrt{1 + |\nabla \phi(x')|^2}} \right) = (p-1)|\nabla u|^p (x', \phi(x')) \tag{13}$$

weakly in $\{|x'| < \delta\}$. The C^α regularity of the right hand side and the theory of quasilinear elliptic equations (see [GT]) give that ϕ is $C^{2,\alpha}$, so the reduced free boundary is $C^{2,\alpha}$-smooth as well and the free boundary condition (11) is true pointwise on Γ^*.

Step 2: Now since higher regularity of the boundary implies the higher regularity of the function u up to the boundary, we can use the bootstrapping argument and obtain arbitrary smoothness, so the boundary is C^∞.

Step 3: The analyticity follows from the theory of elliptic coercive systems (see [KNS]). Here we refer to the paper of Argiolas ([Ar], p. 144), where a similar problem is treated in all details.

Step 4: The inequality (12) is due to the fact that we can carry out the domain variation only in one direction near ∂B_R. □

5. A concavity result

From now on we denote by $\kappa(\partial U)$ the interior mean curvature (in viscosity sense) of the $C^{1,1}$ part of the boundary of a domain U as

follows. Assume $0 \in \partial U$ and the interior normal $\nu_{\partial U}(0)$ shows in the direction of the e-axis. We take

$$\kappa(\partial U)(0) := \inf_{\mathcal{A} \in \mathfrak{A}} \kappa(S_{\mathcal{A}})(0),$$

where $S_{\mathcal{A}} = \{(x, e) | e = \langle \mathcal{A}x, x\rangle\}$ and $\mathfrak{A}$ is the set of all symmetric matrices $\mathcal{A}$ such that the set $S_{\mathcal{A}}$ (the graph of a quadratic polynomial) locally touches ∂U from inside.

Let us consider the convex hull $\mathrm{cov}(U)$ of a (non-convex) set U with C^2 boundary. Note that then $\mathrm{cov}(U)$ has a $C^{1,1}$ boundary (see [KK]). For notational reasons let us assume $U \subset \mathbb{R}^{n+1} = \{(x, e)|x \in \mathbb{R}^n, e \in \mathbb{R}\}$.

The following lemma will be useful and is easy to prove.

Lemma 11. *The function $\kappa(\partial cov(U))(x)$ is upper semi-continuous on $\partial cov(U)$.*

Assume we have a point $x_0 \in \partial\mathrm{cov}(U)\backslash\partial U$, then from the definition of the convex hull we know that x_0 is a convex combination of n points from $\partial\mathrm{cov}(U) \cap \partial U$, i.e., $x_0 = \sum_{k=1}^n \alpha_k y_k$, $\sum_{k=1}^n \alpha_k = 1$, $\alpha_k \geq 0$, $y_k \in \partial\mathrm{cov}(U) \cap \partial U$ for $k = 1, \dots, n$. Since $x_0 \notin \partial U$ more than one of α_k will be different from zero, thus there are points $y_0, z_0 \in \partial\mathrm{cov}(U)$ such that x_0 lies in the interval $(y_0, z_0) \subset \partial\mathrm{cov}(U)$.

Lemma 12. *The function*

$$\frac{1}{\kappa(\partial cov(U))}(x)$$

is concave on the interval $(y_0, z_0) \subset \partial cov(U)$. Moreover if $\kappa(\partial cov(U))(x) = 0$ for some $x \in (y_0, z_0)$ then $\kappa(\partial cov(U))(x) = 0$ for all $x \in (y_0, z_0)$.

Proof. We need to show that

$$\frac{1}{\kappa(\partial\mathrm{cov}(U))}\Big(\frac{x^1 + x^2}{2}\Big) \geq \frac{1}{2}\Big(\frac{1}{\kappa(\partial\mathrm{cov}(U))}(x^1) + \frac{1}{\kappa(\partial\mathrm{cov}(U))}(x^2)\Big)$$

for all $x^1, x^2 \in (y_0, z_0)$. Without loss of generality we can assume $x^1 = (-1, 0, \dots, 0)$ and $x^2 = (1, 0, \dots, 0)$. Since the supporting planes of $\mathrm{cov}(U)$ at x^1 and x^2 coincide we can further assume that the graphs of quadratic polynomials

$$u = \langle \mathcal{A}_1(x - x^1), (x - x^1)\rangle \text{ and } u = \langle \mathcal{A}_2(x - x^2), (x - x^2)\rangle$$

given by positive symmetric matrices $\mathcal{A}_1$ and $\mathcal{A}_2$ locally touch the boundary $\partial\mathrm{cov}(U)$ from inside and $0 < 2\mathrm{Tr}\mathcal{A}_i - \kappa(x^i) < \epsilon$ for $i = 1, 2$.

Since x^1, x^2 lie on the x_1 axis we can assume that for $i = 1, 2$

$$\mathcal{A}_i = \begin{pmatrix} a_i & 0 & \dots & 0 \\ 0 & & & \\ \vdots & & \mathcal{B}_i & \\ 0 & & & \end{pmatrix},$$

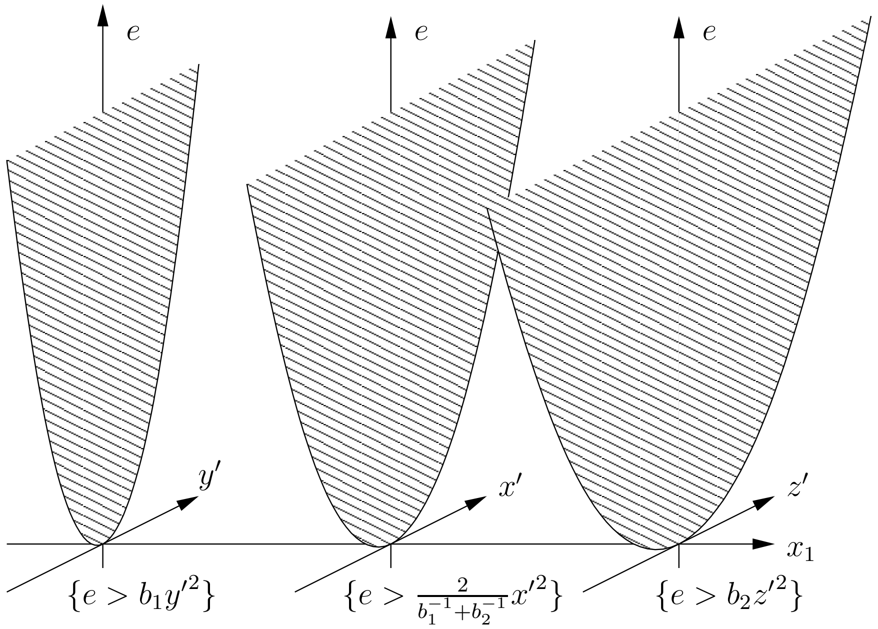

FIGURE 1. Convex hull of two parabolas

where $\mathcal{B}_i$ are positive symmetric matrices and $0 < a_i < \epsilon$. The proof of this fact can be found in the Appendix.

Let us now consider the sets

$$\{(-1, x', e) | e > \langle \mathcal{B}_1 x', x' \rangle\} \text{ and } \{(1, x', e) | e > \langle \mathcal{B}_2 x', x' \rangle\},$$

which touch the boundary of $\mathrm{cov}(U)$ from inside locally at the points x^1 and x^2 respectively. Here $x' = (x_2, \dots, x_n)$. We will now "calculate" the intersection of the convex hull of this two sets with the plane $\{x | x_1 = 0\}$. This will locally touch the boundary $\partial\mathrm{cov}(U)$ from inside and give us the desired estimate on the mean curvature. The intersection of the convex hull of these two sets with the mentioned plane is $\{(0, x', e) | e > u(x')\}$, where

$$u(x') = \inf_{y'+z'=2x'} \frac{1}{2} \left(\langle \mathcal{B}_1 y', y' \rangle + \langle \mathcal{B}_2 z', z' \rangle \right). \tag{14}$$

We are going to calculate explicitly the expression on the right hand side. So for each x' we are looking for the minimum of the following function

$$w_{x'}(y') = \frac{1}{2} (\langle \mathcal{B}_1 y', y' \rangle + \langle \mathcal{B}_2 (2x' - y'), 2x' - y' \rangle).$$

After differentiation in y' and some (simple) calculations we get that the infimum in (14) is attained at the values

$$y' = 2(\mathcal{B}_1 + \mathcal{B}_2)^{-1} \mathcal{B}_2 x'$$

and

$$z' = 2x' - y' = 2(\mathcal{B}_1 + \mathcal{B}_2)^{-1}\mathcal{B}_1 x'.$$

Substituting now the values of y' and z' into (14) and using the identity

$$\mathcal{B}_1(\mathcal{B}_1 + \mathcal{B}_2)^{-1}\mathcal{B}_2 = (\mathcal{B}_1^{-1} + \mathcal{B}_2^{-1})^{-1}$$

we get

$$u(x') = 2\langle(\mathcal{B}_1^{-1} + \mathcal{B}_2^{-1})^{-1}x', x'\rangle.$$

Note that the invertibility of $\mathcal{B}_1 + \mathcal{B}_2$ and $\mathcal{B}_1^{-1} + \mathcal{B}_2^{-1}$ follows from the strict positivity of all eigenvalues of $\mathcal{B}_1, \mathcal{B}_2$. In three dimensions, when matrices $\mathcal{B}_1, \mathcal{B}_2$ are given by positive numbers b_1, b_2, this interesting result is illustrated in Figure 1.

The proof now follows from the inequalities bellow:

$$\begin{aligned}(15)\quad \frac{2}{\kappa(\partial \mathrm{cov}(U))}\Big(\frac{x^1 + x^2}{2}\Big) &\geq \frac{1}{2\mathrm{Tr}(\mathcal{B}_1^{-1} + \mathcal{B}_2^{-1})^{-1}} \geq \frac{1}{2\mathrm{Tr}\mathcal{B}_1} + \frac{1}{2\mathrm{Tr}\mathcal{B}_2} \\ &\geq \frac{1}{\kappa(\partial \mathrm{cov}(U))(x^1) + \epsilon} + \frac{1}{\kappa(\partial \mathrm{cov}(U))(x^2) + \epsilon}.\end{aligned}$$

Note that $\epsilon > 0$ is arbitrary small and we have the first and the third inequalities in (15) by the construction of $\mathcal{B}_1, \mathcal{B}_2$ and from the properties of the convex hull. The second inequality can be found in [ALL].

The case when $\kappa(\partial \mathrm{cov}(U))(x) = 0$ for some $x \in (y_0, z_0)$ follows from (15). □

6. Convexity of the free boundary

In the proof of the key Lemma 14 we will use the following lemma (Lemma 4.1, [LS]). Let $K \subset U$ be a compact convex set, U be open and non-convex and $\mathrm{cov}(U)$ be the convex hull of U. Further assume that the function u minimizes the functional (2) over the set $\{v \in H_0^1(\mathrm{cov}(U)) | v \equiv 1 \text{ on } K\}$ and that the segment $[y_0, z_0] \subset \partial \mathrm{cov}(U)$. Then the following lemma is true.

Lemma 13. *The function*

$$\frac{1}{|\nabla u|}(x)$$

is convex on (y_0, z_0).

This is due to the fact (see [L]) that the level sets of a p-harmonic potential in a convex ring are convex.

The following lemma is key to the proof of the main result.

Lemma 14. *Let u be a (local) minimizer of (2) and denote by $cov(\Omega_u)$ the convex hull of Ω_u. Assume u^c be the minimizer of*

$$(16)\qquad \int_{cov(\Omega_u)\setminus K} |\nabla v(x)|^p dx$$

over the set $\{v \in H_0^1(cov(\Omega_u)) | v \equiv 1 \text{ on } K\}$.

Then $\partial cov(\Omega_u)$ is locally a $C^{1,1}$ surface and is a solution of the (pointwise) free boundary inequality

$$(p-1)|\nabla u^c(x)|^p \geq \kappa(\partial cov(\Omega_u)), \tag{17}$$

where κ is the interior mean curvature.

Proof. The $C^{1,1}$ regularity of the $\partial\mathrm{cov}(\Omega_u)$ follows from the fact that at all points of $\partial\Omega_u \cap \partial\mathrm{cov}(\Omega_u)$ we have a supporting plane, thus (Remark 2) $\partial\Omega_u$ is smooth in the neighborhood of this points, i.e., all singular points of $\partial\Omega_u$ have positive distance from $\partial\mathrm{cov}(\Omega_u)$. This means that $\partial\mathrm{cov}(\Omega_u)$ is as regular as a convex hull of a domain with smooth boundary, that is $C^{1,1}$ (see [KK]).

We get the desired inequality on $\partial\mathrm{cov}(\Omega_u) \cap \partial\Omega_u$ from the maximum principle and Lemma 10.

Assume now that $x_0 \in \partial\mathrm{cov}(\Omega_u)\backslash\partial\Omega_u$. From the definition of the convex hull it follows that we can always write $x_0 = \sum_{k=1}^m \alpha_k y_k$, $y_k \in \partial\mathrm{cov}(\Omega_u) \cap \partial\Omega_u$, $\alpha_k > 0$, $\sum_{k=1}^m \alpha_k = 1$, $2 \leq m \leq n$.

We proceed by induction in m. Assume there exist two points $y_1, y_2 \in \partial\mathrm{cov}(\Omega_u) \cap \partial\Omega_u$ such that y_1, x_0 and y_2 lay on one line.

We need to show that

$$\frac{1}{p-1}\left(\frac{1}{|\nabla u^c(x)|}\right)^p - \frac{1}{\kappa(\partial\mathrm{cov}(\Omega_u))(x_0)} \leq 0. \tag{18}$$

We know that $\frac{1}{|\nabla u^c(x)|}$ and thus $\left(\frac{1}{|\nabla u^c(x)|}\right)^p$ is convex on $[y_1, y_2]$ (Lemma 13). Since (18) is true at the points y_1 and y_2 the proof follows from the concavity of $\frac{1}{\kappa(\partial\mathrm{cov}(\Omega_u))(x)}$ on the line segment (y_1, y_2) and its lower semi-continuity (Lemmas 11 and 12).

The induction step $m \Rightarrow m+1$ finishes the proof. □

Theorem. *If K is convex and u is a minimizer of (2) then Ω_u is also convex.*

Proof. Suppose Ω_u is not convex. Let us take u^c and $\mathrm{cov}(\Omega_u)$ as in Lemma 14 and assume $0 \in \mathrm{int}K$. Further take $u^c_r(x) := u^c(rx)$, $\mathrm{cov}(\Omega^r_u) = r^{-1}\mathrm{cov}(\Omega_u)$ and $r_0 := \inf\{r > 0 | \mathrm{cov}(\Omega^r_u) \subset \Omega_u\} > 1$. Assume $\partial\mathrm{cov}(\Omega^{r_0}_u)$ touches $\partial\Omega_u$ at the point $\tilde{x}$. First note that as in Remark 2 we have that $\tilde{x}$ is not on ∂B_R and that $\partial\Omega_u$ is analytic near $\tilde{x}$. We have now

$$\begin{aligned}\kappa(\partial\mathrm{cov}(\Omega^{r_0}_u))(\tilde{x}) \leq r_0^{1-p}(p-1)|\nabla u^c_{r_0}|^p(\tilde{x}) \leq \\ r_0^{1-p}(p-1)|\nabla u|^p(\tilde{x}) = r_0^{1-p}\kappa(\partial\Omega_u)(\tilde{x}),\end{aligned} \tag{19}$$

where the first inequality follows from Lemma 14, the second one from the comparison principle and the third equality is the free boundary condition. On the other hand from the definition of r_0 we get that $\kappa(\partial\mathrm{cov}(\Omega^{r_0}_u))(\tilde{x}) \geq \kappa(\partial\Omega_u)(\tilde{x})$ and $r_0 > 1$, a contradiction. □

Corollary 15. *The free boundary is an analytic surface.*

Corollary 16. *Using the same method as in the proof of the theorem one can easily prove the uniqueness of the minimizer by a contradiction argument. Note that in the two-phase (interior) case (see* [ACKS]*) the minimizer is not unique.*

Corollary 17. *Again by the same method one can prove that the domain Ω_u of a minimizer u of the problem with general compact set K (even non-connected) is included in the domain $\Omega_{\tilde{u}}$ of the minimizer $\tilde{u}$ of the problem with compact set cov(K).*

Corollary 18. *For large enough R*

$$\Omega_{u_K} \Subset B_R.$$

Proof. Due to the previous corollary we need to prove this only for convex K. If $y \in \partial B_R \cap \Gamma_{u_K}$ then by convexity the conical set $C(y, K) := \{x | x \in [y, z], z \in K\} \subset \Omega_{u_K}$ and

$$cR^2 \leq \mathrm{Per} C(y, K) \leq \mathrm{Per}\Omega_{u_K},$$

where the constant c depends on the set K. This contradicts to the fact that the total energy $I(u)$ should decrease with R. □

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Appendix

Assume $0 \in L \subset \partial\Omega$, where L is the segment connecting $(-1, 0, \dots, 0)$ and $(1, 0, \dots, 0)$, Ω is convex domain with $C^{1,1}$-boundary and $\Omega \subset \{x \in \mathbb{R}^{n+1} | x_{n+1} > 0\}$. We will show here that

$$\kappa(\partial U)(0) := \inf_{\mathcal{A} \in \mathfrak{A}} \kappa(S_{\mathcal{A}})(0) = \inf_{\mathcal{A} \in \mathfrak{A}^*} \kappa(S_{\mathcal{A}})(0)$$

where $S_{\mathcal{A}} = \{(x, e) | e = \langle \mathcal{A}x, x \rangle\}$, $\mathfrak{A}$ is the set of all symmetric matrices $\mathcal{A}$ such that the set $S_{\mathcal{A}}$ (the graph of a quadratic polynomial) locally touches ∂U from inside and $\mathfrak{A}^* \subset \mathfrak{A}$ is the subset of the matrices of the form

$$\mathcal{A} = \begin{pmatrix} a & 0 & \dots & 0 \\ 0 & & & \\ \vdots & & \mathcal{B} & \\ 0 & & & \end{pmatrix}.$$

Note that the boundary of the set Ω around 0 can be locally given as graph of the following function

$$u = f_{x_1}(x') + o(|x'|^2), \text{ as } x' \to 0$$

where $x' = (x_2, \dots, x_n)$ and f_{x_1} are homogeneous functions of order two, i.e., $f_{x_1}(x') = f_{x_1}(\frac{x'}{|x'|})|x'|^2$. It is enough to show that the interior mean curvature of $\partial\Omega$ and of $\{x \in \mathbb{R}^{n+1} | x_{n+1} = f_0(x')\}$ at point 0 is

the same. To see this let as fix any unit vector $e = (\alpha_1, \dots, \alpha_n)$ in $\mathbb{R}^n$ and note that

$$f_{\alpha_1 t}(\alpha_2 t, \dots, \alpha_n t) - f_0(\alpha_2 t, \dots, \alpha_n t) = \\ (f_{\alpha_1 t}(\alpha_2, \dots, \alpha_n) - f_0(\alpha_2, \dots, \alpha_n))t^2 = o(t^2),$$

as $t \to 0+$.

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